Volume 24, Number 4, April 2011

ISSN 0893-9659

ELSEVIER

Applied Mathematics Letters

an international journal of rapid publication

Applied Mathematics Letters 24 (2011) 501–505

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

Janowski starlikeness for a class of analytic functions

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ARTICLE INFO

Article history:
Received 4 June 2009
Received in revised form 29 October 2010
Accepted 2 November 2010

Keywords:
Subordination
Best dominant
Janowski starlike functions
Ma–Minda starlike functions

ABSTRACT

A normalized analytic function f defined on the open unit disk is a Janowski starlike function if $zf'(z)/f(z)$ is subordinated to $(1+Az)/(1+Bz)$, where A and B are complex numbers satisfying the conditions $|B| \leq 1$ and $A \neq B$. In this paper, a new class of analytic functions defined by means of subordination is introduced. Sufficient conditions are obtained for functions in this class to be Janowski starlike. The results obtained extend earlier known works.

1. Introduction and motivation

Let $\mathcal{A}$ be the class of all analytic functions f defined in the open unit disk $\Delta := \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = 0 = f'(0) - 1$. If f and g are analytic in Δ, then f is subordinate to g, written $f(z) \prec g(z)$, if there is an analytic function w, satisfying $w(0) = 0$ and $|w(z)| < 1$, such that $f(z) = g(w(z))$. In case g is univalent in Δ, then f is subordinate to g if and only if $f(0) = g(0)$ and $f(\Delta) \subseteq g(\Delta)$. Let A and B be complex numbers that satisfy the conditions $|B| \leq 1$ and $A \neq B$, and let $S^*[A, B]$ denote the class of *Janowski starlike functions* consisting of $f \in \mathcal{A}$ satisfying the subordination

$$\frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz}.$$

Without loss of generality, it can be assumed that B is real. If A is also real with $|A| \leq 1$, the fact that $S^*[A, B] = S^*[-A, -B]$ permits us to assume that $B < A$. For $-1 \leq B < A \leq 1$, this class was introduced by Janowski and investigated in [1,2].

Several well-known subclasses of starlike functions are special cases of the class $S^*[A, B]$ for suitable choices of the parameters A and B; in particular, when $0 \leq \alpha < 1$, $S^*[1-2\alpha, -1] =: S^*(\alpha)$ is the familiar class of starlike functions of order α. For $A = 1 - 2\beta$, $\beta > 1$ and $B = -1$, denote the class $S^*[1-2\beta, -1]$ by $\mathbb{M}(\beta)$. Equivalently, $\mathbb{M}(\beta)$ can be expressed in the form

$$\mathbb{M}(\beta) := \left\{ f \in \mathcal{A} : \Re\left(\frac{zf'(z)}{f(z)}\right) < \beta, (z \in \Delta) \right\}.$$

The class $\mathbb{M}(\beta)$ was investigated by Uralegaddi et al. [3], while a subclass of $\mathbb{M}(\beta)$ was investigated by Owa and Srivastava [4]. It should be noted that functions in the class $\mathbb{M}(\beta)$ and in general $S^*[A, B]$ need not be starlike. The class $S^*[A, B]$ unifies the

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0893-9659/$ – see front matter
doi:10.1016/j.aml.2010.11.001

classes $S^*(\alpha)$ and $\mathbb{M}(\beta)$; this will not happen if the assumption is only that $-1 \le B < A \le 1$. Ma and Minda [5] have earlier introduced and investigated the class $S^*(\phi)$ of analytic functions $f \in \mathcal{A}$ for which

$$\frac{zf'(z)}{f(z)} \prec \phi(z), \quad z \in \Delta,$$

where ϕ is an analytic function with positive real part on Δ, $\phi(0) = 1$, $\phi'(0) > 0$, and ϕ maps Δ onto a region starlike with respect to 1 and symmetric with respect to the real axis. The class $S^*(\phi)$ contains many of the classes investigated in the literature such as functions that are starlike (of order α), strongly starlike, parabolic starlike, and Janowski starlike (for real constants A and B).

For $0 < \alpha \le 1$, $\lambda > 0$, Tuneski and Irmak [6] introduced and studied the class

$$\mathcal{G}_{\lambda,\alpha} = \left\{ f \in \mathcal{A} : \left| \frac{f(z)}{zf'(z)} \left\{ 1 - \alpha + \alpha \frac{zf''(z)}{f'(z)} \right\} - (1-\alpha) \right| < \lambda, z \in \Delta \right\}.$$

The class $\mathcal{G}_{\lambda,\alpha}$ also includes several other classes investigated earlier, for example,

$$\mathcal{G}_{\lambda,1/2} = \left\{ f \in \mathcal{A} : \left| \frac{f(z)}{zf'(z)} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} - 1 \right| < 2\lambda, z \in \Delta \right\},$$

$$\mathcal{G}_{\lambda,1} = \left\{ f \in \mathcal{A} : \left| \frac{f(z)f''(z)}{(f'(z))^2} \right| < \lambda, \ z \in \Delta \right\},$$

$$\mathcal{G}_{\lambda,1/(2-\gamma)} = \left\{ f \in \mathcal{A} : \left| \frac{f(z)}{zf'(z)} \left\{ 1 - \gamma + \frac{zf''(z)}{f'(z)} \right\} - (1-\gamma) \right| < \lambda(2-\gamma), \ z \in \Delta \right\}.$$

These or related classes were investigated in [7–14].

Using the theory of first-order differential subordination, Tuneski and Irmak [6] and Tuneski [15] obtained the following result of embedding the class $\mathcal{G}_{\lambda,\alpha}$ into the class $S^*[A, B]$.

Theorem 1 (*[6, Theorem 2.2]*). *Let $f \in \mathcal{A}$, $-1 \le B < A \le 1$, and $(1 + |A|)/(3 + |A|) \le \alpha \le 1$. If*

$$\frac{f(z)}{zf'(z)} \left\{ 1 - \alpha + \alpha \frac{zf''(z)}{f'(z)} \right\} \prec \alpha + (1 - 2\alpha) \frac{1 + Bz}{1 + Az} + \alpha \frac{z(A - B)}{(1 + Az)^2},$$

then $f \in S^*[A, B]$. This result is sharp.

As a consequence, the following result is obtained:

Corollary 1 (*[6, Corollary 2.4]*). *Let $-1 \le B < A \le 1$ and $(1 + |A|)/(3 + |A|) \le \alpha \le 1$. Then*

$$\lambda = (A - B) \frac{(2\alpha - 1)|A| - (1 - 3\alpha)}{(1 + |A|)^2} \tag{1.1}$$

is the greatest number such that $\mathcal{G}_{\lambda,\alpha} \subseteq S^*[A, B]$.

Note that there was a typographic error in sign in the work of [6], and that expression (1.1) is the correct constant.

We now introduce a class of analytic functions defined by means of subordination.

Definition 1. For complex constants C and D with $|D| \le 1$, $C \ne D$, the class $\mathcal{G}_\alpha[C, D]$ consists of all functions $f \in \mathcal{A}$ satisfying the subordination

$$\frac{f(z)}{zf'(z)} \left\{ 1 - \alpha + \alpha \frac{zf''(z)}{f'(z)} \right\} \prec (1 - \alpha) \frac{1 + Cz}{1 + Dz}.$$

For $0 < \alpha \le 1$, $\lambda > 0$, the class $\mathcal{G}_\alpha[\lambda/(1 - \alpha), 0]$ reduced to the class $\mathcal{G}_{\lambda,\alpha}$ studied by Tuneski and Irmak [6]. In this paper, we investigate the more general inclusion $\mathcal{G}_\alpha[C, D] \subseteq S^*[A, B]$. The following result will be required.

Theorem 2 (*[16, Theorem 3.4h, p.132]*). *Let q be univalent in the unit disk Δ and ϑ and φ be analytic in a domain D containing $q(\Delta)$ with $\varphi(w) \ne 0$ when $w \in q(\Delta)$. Set $Q(z) := zq'(z)\varphi(q(z))$ and $h(z) := \vartheta(q(z)) + Q(z)$. Suppose that either h is convex, or Q is starlike univalent in Δ. In addition, assume that $\Re[zh'(z)/Q(z)] > 0$ for $z \in \Delta$. If p is analytic in Δ with $p(0) = q(0)$, $p(\Delta) \subseteq D$ and*

$$\vartheta(p(z)) + zp'(z)\varphi(p(z)) \prec \vartheta(q(z)) + zq'(z)\varphi(q(z)), \tag{1.2}$$

then $p(z) \prec q(z)$ and q is the best dominant.

2. Main results

We begin with the following sufficient condition for a function $f \in \mathcal{A}$ to satisfy the subordination $zf'(z)/f(z) \prec 1/q(z)$.

Theorem 3. *Let α be a nonzero complex number. Let q be univalent and $q(z) \neq 0$ in Δ, $q(0) = 1$ and*

$$\Re\left\{1 + \frac{zq''(z)}{q'(z)}\right\} > \max\left\{0, \Re\left(\frac{1-2\alpha}{\alpha}\right)\right\}. \tag{2.1}$$

If $f \in \mathcal{A}$ satisfies the subordination

$$\frac{f(z)}{zf'(z)}\left\{1 - \alpha + \alpha\frac{zf''(z)}{f'(z)}\right\} \prec \alpha + (1-2\alpha)q(z) - \alpha zq'(z), \tag{2.2}$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{1}{q(z)}$$

and $1/q$ is the best dominant.

Proof. Let the function p be defined by

$$p(z) = \frac{f(z)}{zf'(z)}. \tag{2.3}$$

A computation from (2.3) gives

$$\frac{zp'(z)}{p(z)} = \frac{zf'(z)}{f(z)} - \left(1 + \frac{zf''(z)}{f'(z)}\right),$$

and hence

$$1 + \frac{zf''(z)}{f'(z)} = -\frac{zp'(z)}{p(z)} + \frac{1}{p(z)}. \tag{2.4}$$

Now (2.3) and (2.4) yield

$$\frac{f(z)}{zf'(z)}\left\{1 - \alpha + \alpha\frac{zf''(z)}{f'(z)}\right\} = \alpha + (1-2\alpha)p(z) - \alpha zp'(z). \tag{2.5}$$

Using (2.5), it follows that (2.2) becomes

$$\alpha + (1-2\alpha)p(z) - \alpha zp'(z) \prec \alpha + (1-2\alpha)q(z) - \alpha zq'(z),$$

or

$$(1-2\alpha)p(z) - \alpha zp'(z) \prec (1-2\alpha)q(z) - \alpha zq'(z). \tag{2.6}$$

Define the functions ϑ and φ by

$$\vartheta(w) = (1-2\alpha)w, \qquad \varphi(w) = -\alpha$$

so that (2.6) becomes (1.2). Since $\alpha \neq 0$, clearly $\varphi(w) \neq 0$. Now let

$$Q(z) := zq'(z)\varphi(q(z)) = -\alpha zq'(z)$$
$$h(z) := \vartheta(q(z)) + Q(z) = (1-2\alpha)q(z) - \alpha zq'(z).$$

In view of (2.1), Q is starlike and

$$\Re\left[\frac{zh'(z)}{Q(z)}\right] = \Re\left\{1 + \frac{zq''}{q'} - \frac{1-2\alpha}{\alpha}\right\} > 0.$$

The result now follows by an application of Theorem 2. $\square$

Corollary 2. *Let $\alpha \in \mathbb{C}$, $-1 \leq B < A \leq 1$, and further assume that*

$$\Re\left(\frac{1}{\alpha}\right) \leq \frac{3 + |A|}{1 + |A|}. \tag{2.7}$$

If $f \in \mathcal{A}$ satisfies

$$\frac{f(z)}{zf'(z)}\left\{1-\alpha+\alpha\frac{zf''(z)}{f'(z)}\right\} \prec \alpha+(1-2\alpha)\frac{1+Bz}{1+Az}+\alpha\frac{z(A-B)}{(1+Az)^2},$$

then $f \in S^*[A, B]$. The result is sharp.

Proof. Let the function q be defined by $q(z) = (1 + Bz)/(1 + Az)$. This function q is convex univalent and (2.7) yields

$$\Re\left(1+\frac{zq''(z)}{q'(z)}\right)=\Re\left(\frac{1-Az}{1+Az}\right)>\frac{1-|A|}{1+|A|}\geq \max\left(0,\Re\frac{1-2\alpha}{\alpha}\right).$$

The result now follows from Theorem 3. □

Remark 1. When α is real, Corollary 2 reduces to Theorem 1.

Theorem 4. *Let A, B, C, D and α be real numbers satisfying $|D| \leq 1$, $C \neq D$, $|A| \leq 1$, $|B| \leq 1$, $A \neq B$ and $\alpha \neq 0$. Let $I := (3\alpha-1)^2+(2\alpha-1)^2A^2$, $J := 2(3\alpha-1)(2\alpha-1)A$, $K := (C-D)(1-\alpha)$, $L := A^2C(1-\alpha)-AD\alpha(A-2B)-ABD$, and $M := 2AC(1-\alpha)-D(A+B)-D\alpha(A-3B)$. Further, when $KL < 0$ and $|(A-B)^2J-2(K+L)M| < -8KL$, assume that*

$$-16KL[(A-B)^2I-M^2(L-K)^2]-[(A-B)^2J-2(K+L)M]^2\geq 0$$

while in all other cases, let

$$|(A-B)^2J-2(K+L)M|\leq (A-B)^2I-M^2-(L+K)^2.$$

Then $\mathcal{G}_\alpha[C, D] \subseteq S^*[A, B]$.

Proof. In view of Theorem 3, it is enough to show that

$$g(z):=(1-\alpha)\frac{1+Cz}{1+Dz}\prec \alpha+(1-2\alpha)\frac{(1+Bz)}{(1+Az)}+\frac{\alpha z(A-B)}{(1+Az)^2}=:h(z).$$

Since g is univalent, the subordination $g(z) \prec h(z)$ is equivalent to the subordination

$$z\prec g^{-1}(h(z))=:H(z).$$

The proof will be completed by showing that $|H(e^{i\theta})| \geq 1$ for all $\theta \in [0, 2\pi]$. First note that

$$h(z)=\frac{1-\alpha+[A+B+\alpha(A-3B)]z+[A\alpha(A-2B)+AB]z^2}{(1+Az)^2},$$

and

$$g^{-1}(w)=\frac{w+\alpha-1}{C(1-\alpha)-Dw},$$

so that

$$H(z)=\frac{(A-B)[(3\alpha-1)+(2\alpha-1)Az]z}{K+Mz+Lz^2}.$$

Writing $t = \cos\theta$, it follows that

$$\begin{aligned}|H(e^{i\theta})|^2 &= \frac{(I+Jt)(A-B)^2}{|Ke^{-i\theta}+M+Le^{i\theta}|^2}\\ &= \frac{(I+Jt)(A-B)^2}{4KLt^2+2(K+L)Mt+M^2+(L-K)^2}.\end{aligned}$$

Now $|H(e^{i\theta})|^2 \geq 1$ provided $at^2+bt+c \geq 0$, where $a = -4KL$, $b = (A-B)^2J-2(K+L)M$, and $c = (A-B)^2I-M^2-(L-K)^2$. Since

$$\min_{|t|\leq 1}\{at^2+bt+c\}=\begin{cases}\dfrac{4ac-b^2}{4a}, & a>0,\ |b|<2a,\\ a+c-|b|, & \text{otherwise,}\end{cases}$$

the inequality $|H(e^{i\theta})| \geq 1$ is satisfied provided the conditions stated in Theorem 4 hold. □

Remark 2. When $D = 0$, $C = \lambda/(1-\alpha)$, we have $I = (3\alpha-1)^2 + (2\alpha-1)^2 A^2$, $J = 2(3\alpha-1)(2\alpha-1)A$, $K = \lambda$, $L = \lambda A^2$, $M = 2A\lambda$. Clearly $KL = \lambda^2 A^2 \geq 0$. In this case, the condition in the hypothesis of Theorem 4 becomes

$$|(A-B)^2 J - 4A\lambda^2(1+A^2)| \leq (A-B)^2 I - 4A^2\lambda^2 - \lambda^2[4A^2 + (A^2-1)^2].$$

A computation shows that

$$\lambda = (A-B)\frac{(2\alpha-1)|A| - (1-3\alpha)}{(1+|A|)^2}$$

provided $(1+|A|)/(3+|A|) \leq \alpha < 1$. Thus Theorem 4 reduces to [6, Corollary 2.4, p. 4].

Remark 3. In [17] and [18] a similar technique using Jack's lemma was used to investigate Janowski starlikeness of the Bernardi integral operator.

Acknowledgements

The work presented here was supported in part by research grants from Universiti Sains Malaysia and University of Delhi.

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